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A DELAMINATION MODEL FOR LAMINATED COMPOSITES

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Abstract—The delamination problem of laminated composite plates is considered. The Frémond's adhesion approach is developed and adapted to the delamination problem considered. A consistent thermodynamical formulation for the damage model is presented and the governing equations are carried out. The relation between the proposed approach and the fracture mechanics theory is emphasized. Furthermore, a regularized model is developed. A numerical procedure based on the finite element method and on the elastic predictor–damage corrector method is proposed. Numerical results carried out for beams are compared with the analytical solutions. Finally, the problem of drilling a composite laminate is investigated.

1. INTRODUCTION

Laminated composites are made by several thin laminae bonded together to act as an integral structural element. Imperfection of adhesion between laminae may be introduced in the laminate during the fabrication process or manufacturing operations. Delamination, that is growth in size and number of interlaminar defects, can occur when the laminate is subjected to accidental loading applied under quasi-static or dynamic (impact) mode. The delamination leads to a loss of stiffness and strength of the laminate. The effects of delamination can be ruinous, since the evolution of the interlaminar defects can carry the structure to total failure followed by collapse.

The literature regarding the delamination and related problems is large. A presentation of several structural problems of delamination, with up-dated extensive references, can be found in Garg (1988) and Ochoa and Reddy (1992). Delamination may be caused by interlaminar stress concentration occurring in the neighborhood of the free edge or around loaded holes of the composite laminated plates (Jain and Yang, 1991). Furthermore, interlaminar defects may grow when the laminate is subjected to compressive loads. In this case a thin laminated layer buckles, causing intense stresses at the delamination boundary, and a global structure failure at loads below the design level occurs (Bottega and Maewal, 1983; Chai and Babcock, 1985; Kachanov, 1988; Bruno and Grimaldi, 1990).

In order to determine the load carrying capacity and service effectiveness of the structure, an accurate analysis able to predict the initiation and evolution of damage is needed. The study of the delamination of composite plates may be carried out by adopting a fracture mechanics approach or by introducing appropriate interface constitutive laws between the laminae constituting the whole laminate. From a physical point of view, it seems reasonable to suppose that this second approach can be related to the fracture mechanics approach. In fact, when separation between two bonded solids occurs, there is an evolution of the unglued area that is equivalent to the propagation of a fracture in an *a priori* known direction.

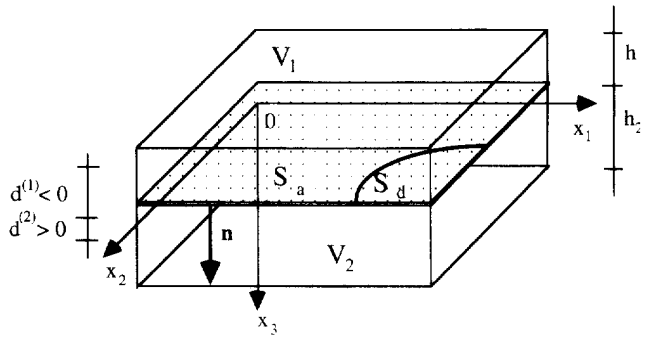

Fig. 1. Geometry of two superimposed plates.

Two- or three-dimensional analyses have been proposed to define the stress distribution around the delaminated area in laminated composite plates. While the two-dimensional approach in some cases appears unable to lead to a satisfactory stress analysis, the three-dimensional approach is always complex and needs strong computational efforts.

The main purpose of this paper is to present a new and comprehensive delamination model which allows to recover either the classical fracture mechanics theory or, via regularization of the involved functionals, a wide variety of the interface constitutive models. A further purpose of the paper is to develop a simple and effective numerical procedure for the analysis of the delamination phenomenon.

With this aim, the delamination problem of a laminate is treated by assuming that only one delamination surface is present in the thickness of the composite plate. In this case the analysis can be carried out by considering two superimposed anisotropic composite plates connected by a special interface adhesive bond. The adopted plate model is introduced. An interface constitutive law, based on the adhesion model proposed in Frémond (1985, 1987, 1988), Point (1989) and Truong Dinh Tien (1990) for the study of a glued surface, is presented. The state equations and the evolution law for the interface are carried out in the formal framework of thermodynamics. Remarks are developed in the presentation of the governing equations. Correlations between the present approach with the fracture mechanics approach are emphasized. Then, a regularization of the model is presented. The interface model proposed in Ascione and Bruno (1985) and Grimaldi and Reddy (1985) is obtained as a special case of the proposed regularized model. Furthermore, the elastic-damage interface model described in Ladevèze (1992) and Allix and Ladevèze (1992), and generalized in Corigliano (1993) can be carried out in the framework of the proposed regularized model.

Finite element formulation and a numerical iterative procedure based on the predictor–corrector method are developed. Computational hints are given for a satisfactory numerical determination of the stiffness matrix of the interface elements.

Finally, numerical results relative to simple structural problems are presented. Beams and plates problems are treated. Analytical solutions carried out for the beam problem are compared with those obtained by means of the finite element formulation. A real technical plate problem is treated. It concerns the delamination induced in a cross-ply laminate during the drilling.

2. THE PLATE MODEL

Let V be a composite laminated plate obtained by superimposing two plates V_1 and V_2 with mid-planes Ω_1 and Ω_2 and constant thicknesses h_1 and h_2, respectively. The surfaces in contact are denoted by $S^{(1)}$ and $S^{(2)}$. The common area of the two plates is denoted by S. A perfect bond between the two plates is considered on $S_a \subset S$ and an initial defect of adhesion is present on $S_d \subset S$. A global cartesian coordinate system (x_1, x_2, x_3) is introduced in the mid-plane of the whole body V, as shown in Fig. 1. The unit vector $\mathbf{n}$ normal to the surface Ω defining the positive direction of the axis x_3 is introduced.

The first-order shear deformation plate theory (Reddy, 1984), that is the Mindlin–Reissner plate theory, is considered to model the behavior of each plate constituting the whole body. Hence, the kinematics is governed by the following displacement vectors:

$$\mathbf{u}^{(*)}(x_1, x_2, x_3; t) = \mathbf{v}^{(*)}(x_1, x_2; t) + [x_3 - d^{(*)}]\boldsymbol{\varphi}^{(*)}(x_1, x_2; t), \tag{1}$$

where superscript $(*)$ assumes value (1) or (2) and refers quantities to the plates V_1 or V_2. The independent variable t represents an evolution parameter which can be identified, for instance, with the time. The vectors $\mathbf{v}^{(*)}$ and $\boldsymbol{\varphi}^{(*)}$ represent the displacement of the mid-plane and the rotation of the fibers normal to the mid-plane of the plates. The values $d^{(1)}$ and $d^{(2)}$ are defined as the coordinates along the x_3-axis of the mid-planes of the plates V_1 and V_2, respectively.

The infinitesimal strain tensor $\mathbf{D}^{(*)} = \hat{\mathbf{D}}^{(*)} + (x_3 - d^{(*)})\tilde{\mathbf{D}}^{(*)}$ has components:

$$\hat{D}^{(*)}_{11} = v^{(*)}_{1,1} \quad \hat{D}^{(*)}_{22} = v^{(*)}_{2,2} \quad \hat{D}^{(*)}_{33} = 0$$

$$2\hat{D}^{(*)}_{23} = \varphi_2 + v^{(*)}_{3,2} \quad 2\hat{D}^{(*)}_{13} = \varphi_1 + v^{(*)}_{3,1} \quad 2\hat{D}^{(*)}_{12} = v^{(*)}_{1,2} + v^{(*)}_{2,1} \tag{2a}$$

$$\tilde{D}^{(*)}_{11} = \varphi^{(*)}_{1,1} \quad \tilde{D}^{(*)}_{22} = \varphi^{(*)}_{2,2} \quad \tilde{D}^{(*)}_{33} = 0 2\tilde{D}^{(*)}_{23} = 0 \quad 2\tilde{D}^{(*)}_{13} = 0 \quad 2\tilde{D}^{(*)}_{12} = \varphi^{(*)}_{1,2} + \varphi^{(*)}_{2,1}. \tag{2b}$$

At this stage, it is worth noting that the plates constituting the whole body are independent. In fact constraints on the relative displacements have not been introduced so far.

From a constitutive point of view, each plate is considered to be composed by several thin orthotropic laminae, with material axes arbitrarily oriented in the x_1–x_2 plane. The linear stress–strain relationship $\mathbf{S} = C[\mathbf{D}]$ in a typical point of the laminate is considered, where $\mathbf{S}$ is the symmetric Cauchy stress tensor and C is the fourth-order elasticity tensor. The extensional, coupling and bending elastic tensors of each laminate constituting the whole body, computed with respect to the coordinate system, are defined as:

$$A^{(*)} = \int_{h_*} C\,\mathrm{d}x_3 \quad B^{(*)} = \int_{h_*} (x_3 - d^{(*)})C\,\mathrm{d}x_3 \quad D^{(*)} = \int_{h_*} (x_3 - d^{(*)})^2 C\,\mathrm{d}x_3. \tag{3}$$

The elastic strain energy of each plate V_* is given by the following symmetric bilinear form:

$$\begin{aligned} \tfrac{1}{2}a^{(*)}(\mathbf{u}^{(*)}, \mathbf{u}^{(*)}) = \frac{1}{2}\int_{V_*} C[\mathbf{D}^{(*)}(\mathbf{u}^{(*)})] \cdot \mathbf{D}^{(*)}(\mathbf{u}^{(*)})\,\mathrm{d}v = \int_{\Omega_*} B^{(*)}[\tilde{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})] \cdot \hat{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})\,\mathrm{d}s \\ - \frac{1}{2}\int_{\Omega_*} A^{(*)}[\hat{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})] \cdot \hat{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})\,\mathrm{d}s - \frac{1}{2}\int_{\Omega_*} D^{(*)}[\hat{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})] \cdot \hat{\mathbf{D}}^{(*)}(\mathbf{u}^{(*)})\,\mathrm{d}s. \end{aligned} \tag{4}$$

The potential energy of the conservative external load for the plate V_* is given by the linear form:

$$f^{(*)}(\mathbf{u}^{(*)}) = \int_{\Omega_*} \mathbf{p}^{(*)} \cdot \mathbf{u}^{(*)}\,\mathrm{d}r, \tag{5}$$

where $\mathbf{p}^{(*)}$ is the distributed load acting on the mid-planes of each plate V_*.

3. THE ADHESION MODEL

Let $P^{(1)} \in S^{(1)}$ be a typical point of the plate V_1 defined by the in-plane coordinates (x_1, x_2) and $P^{(2)} \in S^{(2)}$ the corresponding point of the plate V_2 with the same in-plane coordinates (x_1, x_2). The displacement vectors of the points $P^{(1)}$ and $P^{(2)}$ are:

$$\bar{\mathbf{u}}^{(1)}(x_1,x_2) = \mathbf{v}^{(1)}(x_1,x_2)+\tfrac{1}{2}h_1\boldsymbol{\varphi}^{(1)}(x_1,x_2) \tag{6a}$$

$$\bar{\mathbf{u}}^{(2)}(x_1,x_2) = \mathbf{v}^{(2)}(x_1,x_2)-\tfrac{1}{2}h_2\boldsymbol{\varphi}^{(2)}(x_1,x_2), \tag{6b}$$

where the dependence on the time is understood. The relative displacement vector between the two points under consideration is:

$$\mathbf{s} = s_n\mathbf{n}+\mathbf{s}^t = \bar{\mathbf{u}}^{(2)}-\bar{\mathbf{u}}^{(1)}, \tag{7}$$

where s_n represents the component of $\mathbf{s}$ in the $\mathbf{n}$-direction and $\mathbf{s}^t$ the tangential relative displacement vector.

The interface law proposed herein to describe the behavior of the two plates in contact by means of the surfaces $S^{(1)}$ and $S^{(2)}$, is based on the model developed in Frémond (1985, 1987, 1988) and Point (1989). According to Frémond's model the kinematical variable $\mathbf{s}$, that is the relative displacement vector introduced by eqn (4), is not sufficient to model the contact with adhesion phenomenon. A new variable γ needs to be defined on the interface S. It accounts for the intensity of adhesion, i.e. the proportion of active links between the two typical points of the surfaces in contact of the plates. In other words, γ represents the damage variable and is set to assume the following values:

$$\begin{aligned} \gamma &= 0 \quad \text{when total adhesion is present,} \\ 0 < \gamma &< 1 \quad \text{when the adhesion is partial,} \\ \gamma &= 1 \quad \text{when no adhesion exists.} \end{aligned} \tag{8}$$

Hence, the interface state is defined by means of the couple $(\mathbf{s},\gamma)$. Note that the damage variable introduced herein is not the same as that proposed by Frémond. In fact, his variable named β is related to γ by means of the simple relation: $\beta = 1-\gamma$. The present choice of the adhesion parameter is consistent with the possibility of understanding γ as a damage measure (Kachanov, 1986).

From a physical point of view a very simple behavior of the interface is desired. The main idea consists in considering an on/off, i.e. glued/unglued, character of the interface, which describes the fundamental feature of the adhesion. As a consequence, when adhesion exists ($0 \leqslant \gamma < 1$) between the points $P^{(1)}$ and $P^{(2)}$, the relative displacement vector is constrained to be $\mathbf{s} = \mathbf{0}$. On the contrary, if adhesion is not present ($\gamma = 1$) then the normal component s_n of the relative displacement vector must satisfy the impenetrability condition $s_n \geqslant 0$ and the tangential relative displacement vector remains free to assume any value and direction in the interface plane. Hence, the displacement variable $\mathbf{s}$ is constrained to satisfy the following conditions:

$$\begin{aligned} \mathbf{s} &= \mathbf{0} \quad \text{when there is adhesion:} \quad 0 \leqslant \gamma < 1, \\ s_n &\geqslant 0 \quad \text{when there is no adhesion:} \quad \gamma = 1. \end{aligned} \tag{9}$$

Finally, the displacement vector and the damage variable must satisfy the compatibility conditions:

$$(1-\gamma)\mathbf{s} = \mathbf{0} \quad 0 \leqslant \gamma \leqslant 1 \quad s_n \geqslant 0. \tag{10}$$

Introducing the sets K and Q as:

$$K = \{(\mathbf{s}, \gamma) : \quad (1-\gamma)\mathbf{s} = \mathbf{0} \quad s_n \geqslant 0\} \tag{11}$$

$$Q = \{\gamma : \quad 0 \leqslant \gamma \leqslant 1\} \tag{12}$$

the conditions (10) can be written in the equivalent forms:

$$(\mathbf{s}, \gamma) \text{ admissible} \Leftrightarrow I_K(\mathbf{s}, \gamma) + I_Q(\gamma) = 0. \tag{13}$$

where I_K and I_Q are the indicator functions of the non-convex set K and the convex set Q, respectively. It is worth noting that although the set K is not convex, its indicator function I_K is subdifferentiable, as pointed out in Frémond (1987).

4. STATE EQUATIONS AND EVOLUTION LAW

Once defined, the physics of the adhesion model, the state equations and the evolution law are carried out in the formal framework of the thermodynamics, ensuring the consistency of the model.

The couple $(\mathbf{s}, \gamma)$ represents the set of the state variables for the ideal material present at the interface S, between the two plates. The free energy density $\psi(\mathbf{s}, \gamma)$ for this very special interface material is assumed to be:

$$\psi(\mathbf{s}, \gamma) = I_K(\mathbf{s}, \gamma). \tag{14}$$

Thus, the thermodynamical forces $(\mathbf{t}^R, Y)$ associated with the state variables $(\mathbf{s}, \gamma)$ are defined by:

$$\mathbf{t}^R \in \partial_{\mathbf{s}} \psi(\mathbf{s}, \gamma) = \partial_{\mathbf{s}} I_K(\mathbf{s}, \gamma) \tag{15a}$$

$$Y \in -\partial_{\gamma} \psi(\mathbf{s}, \gamma) = -\partial_{\gamma} I_K(\mathbf{s}, \gamma). \tag{15b}$$

where the symbol $\partial_{(\cdot)}$ indicates the subdifferential operator with the respect to the variable $(\cdot)$. The thermodynamical forces $\mathbf{t}^R$ and Y represent the dual quantities of the state variables. Thus $\mathbf{t}^R$ is a force per unit of area vector working for the relative displacement vector $\mathbf{s}$, hence it represents a reversible stress at the interface. The force Y works for the damage parameter γ, then it is an energy of adhesion per unit area present at the interface.

The introduction of the free energy density (14) allows the determination of the state eqns (15), but is not able to completely define the behavior of the interface model. A complementary law is needed in order to determine the evolution behavior for the rate of the state variables $(\dot{\mathbf{s}}, \dot{\gamma})$. From the thermodynamical point of view, it is fundamental that the evolution law satisfies the Clausius–Duhem inequality, a direct consequence of the second principle of the thermodynamics. In the present context, since variations of temperature are not considered, the Clausius–Duhem relationship is simplified in the form:

$$d = \mathbf{t} \cdot \dot{\mathbf{s}} - \dot{\psi}(\dot{\mathbf{s}}, \dot{\gamma}) \geqslant 0. \tag{16}$$

where d is the so-called dissipation, $\mathbf{t}$ is the stress at the interface and the superscript point indicates the derivative with respect to the time, or any other evolution parameter. Because of eqns (15), the relation (16) becomes:

$$d = (\mathbf{t} - \mathbf{t}^R) \cdot \dot{\mathbf{s}} - Y\dot{\gamma} = \mathbf{t}^{IR} \cdot \dot{\mathbf{s}} + Y\dot{\gamma} \geqslant 0. \tag{17}$$

The total interface stress vectors $\mathbf{t} = \mathbf{t}^R + \mathbf{t}^{IR}$ is obtained as sum of a reversible part $\mathbf{t}^R$ obtained from eqn (15a), and an irreversible one $\mathbf{t}^{IR}$.

In order to satisfy the thermodynamical requirement expressed by eqn (17), the normality hypothesis is introduced. It consists in the assumption of the existence of a convex

positive function called potential of dissipation $\Lambda(\dot{\mathbf{s}}, \dot{\gamma})$, such that $\Lambda(\mathbf{0}, 0) = 0$, or equivalently the existence of its Fenchel's conjugate $\Lambda^*(\mathbf{t}^{IR}, Y)$, even convex and positive with $\Lambda^*(\mathbf{0}, 0) = 0$, governing the evolution behavior by means of the relations:

$$\mathbf{t}^{IR} \in \partial_{\dot{\mathbf{s}}} \Lambda(\dot{\mathbf{s}}, \dot{\gamma}) \tag{18a}$$

$$Y \in \partial_{\dot{\gamma}} \Lambda(\dot{\mathbf{s}}, \dot{\gamma}) \tag{18b}$$

or equivalently

$$\dot{\mathbf{s}} \in \partial_{\mathbf{t}}^{IR} \Lambda^*(\mathbf{t}^{IR}, Y) \tag{19a}$$

$$\dot{\gamma} \in \partial_{Y} \Lambda^*(\mathbf{t}^{IR}, Y). \tag{19b}$$

It can be proved, in a general framework, that the existence of a potential of dissipation and its conjugate with the relations (18) and (19) ensures the validity of (17).

A very simple interpretation of the delamination phenomenon suggests that once a part of the interface is delaminated then it cannot be rebonded. On the other hand, it can be supposed that, after any loading cycle starting from an unstressed condition and inducing delamination, the stress at the interface returns to zero. This means that the interface behaves elastically with respect to the stress or, in other words, the irreversible stress is always zero. As a consequence, the irreversibility of the phenomenon depends only on the variation of the delamination area and hence on the damage parameter. Furthermore, the physics shows that the damage cannot decrease during any loading/unloading cycle.

Now, in order to define a mathematical model satisfying these simple features, a very special form for the conjugate of the potential of dissipation is chosen. Let the quantity $\bar{Y}$ be introduced as:

$$\bar{Y}(t) = \max \{\omega, \sup_{\tau \leqslant t} Y(\tau)\}, \tag{20}$$

where ω is the Dupré's energy of adhesion, that is the energy per unit of area necessary to separate two perfectly bonded surfaces. The convex set $W(t) = (-\infty, \bar{Y}(t)]$ is defined at each time t as the set of all the possible values for Y. The conjugate potential of dissipation is assumed to be the indicator function of the convex set $W(t)$:

$$\Lambda^*(\mathbf{t}^{IR}, Y) = I_W(Y), \tag{21}$$

where the time dependence is understood. It is worth noting that the chosen potential (21) does not depend on the irreversible part of the interface stress vector $\mathbf{t}^{IR}$. As a consequence, the Fenchel's conjugate Λ^{**} of Λ^* does not depend on $\dot{\mathbf{s}}$. It is simple to verify that in the present case the equality $\Lambda^{**} = \Lambda$ holds. Since Λ depends only on $\dot{\gamma}$, no dissipation is related to the rate of relative displacement vector $\dot{\mathbf{s}}$. Hence the irreversible part of the contact stress vector is zero, i.e. $\mathbf{t}^{IR} = \mathbf{0}$, as desired. Thus, the reversible part of the stress given by the formula (15a) represents the total stress at interface, that is $\mathbf{t} = \mathbf{t}^{R}$. On the other hand, as previously pointed out, any admissible adhesion energy Y, for the definition of the quantity $\bar{Y}$, must be equal to or less than $\bar{Y}$. Thus, any admissible adhesion energy satisfies the condition $I_W(Y) = 0$.

The normality rule (19), for the special form of the potential of dissipation (21), assumes the expression:

$$\dot{\gamma} \in \partial I_W(Y). \tag{22}$$

Equation (22) represents the evolution law for the rate of the damage parameter. It can be rewritten in explicit form as:

$$\dot{\gamma} = 0 \quad \text{if } Y < \bar{Y} \tag{23a}$$

$$\dot{\gamma} \geqslant 0 \quad \text{if } Y = \bar{Y}. \tag{23b}$$

Finally, the evolution law obtained is equivalent to the complementarity conditions:

$$Y - \bar{Y} \leqslant 0 \quad (Y - \bar{Y})\dot{\gamma} = 0 \quad \dot{\gamma} \geqslant 0. \tag{24}$$

Once defined the evolution law, it is necessary to check the consistency of the evolution model with respect to the dissipation inequality (17). Since $\mathbf{t}^{IR} = \mathbf{0}$, eqn (17) is simplified in the form:

$$d = Y\dot{\gamma} \geqslant 0 \tag{25}$$

Because of the Fenchel's relationship between conjugate functions Λ and Λ^*, eqns (18) and (19) are equivalent to:

$$Y\dot{\gamma} = \Lambda(\dot{\gamma}) + \Lambda^*(Y). \tag{26}$$

Since both Λ and Λ^* are not negative, eqn (25) is satisfied and the Clausius–Duhem inequality is ensured.

It should be emphasized that the damage parameter, according to the present formulation, cannot decrease. This fact describes the mechanical delamination phenomenon well. Thus, once delamination occurs, there is no possibility of rebonding of the two surfaces. On the contrary, no limitations on the sign of the rate of damage parameter was introduced into the original Frémond's model, because his model simulates the different phenomenon of the adhesion of the two glued surfaces, as, for instance, the problem of the adhesive tape which can be removed from a surface and then rebonded on the surface.

Note that the general formulation presented allows one to define several other state and evolution laws. For instance, plastic or viscous behavior of the interface can be modeled by assuming different forms for the governing functionals.

5. THE GOVERNING EQUATIONS

Let γ_0 be the initial value of the damage parameter in a typical point of the surface S. The range of admissibility for the damage parameter γ is restricted to:

$$\gamma_0 \leqslant \gamma \leqslant 1 \tag{27}$$

as a direct consequence of the evolution condition (23).

The governing equations are carried out using a variational formulation, as proposed in Bottega (1983). In that paper neither the damage parameter was introduced, nor was an evolution law specified. Thus, to have a physically consistent model and a justifiable energy approach, a monotone delamination was assumed. On the contrary, because of thermodynamical framework, in the present model the governing equations are derived without any further assumption.

The total potential energy π of the whole system is given as:

$$\pi(\mathbf{u}^{(1)},\mathbf{u}^{(2)},\mathbf{s},\mathbf{r},\gamma) = \tfrac{1}{2}a^{(1)}(\mathbf{u}^{(1)},\mathbf{u}^{(1)}) + \tfrac{1}{2}a^{(2)}(\mathbf{u}^{(2)},\mathbf{u}^{(2)}) - f^{(1)}(\mathbf{u}^{(1)}) - f^{(2)}(\mathbf{u}^{(2)}) + \int_S I_k(\mathbf{s},\gamma)\,ds - \int_S (1-\gamma)\omega\,ds + \int_S I_A(\gamma)\,ds + \int_S \mathbf{r}\cdot[\mathbf{s} - (\bar{\mathbf{u}}^{(2)} - \bar{\mathbf{u}}^{(1)})]\,ds, \tag{28}$$

where I_A is the indicator function of the set $A = \{\gamma : \gamma_0 \leqslant \gamma \leqslant 1\}$. The vector $\mathbf{r}$ is the Lagrange multiplier for the constraint (7). The functional (28) is equal to $+\infty$ if $(\mathbf{s},\gamma)$ does not belong

to the set K or γ does not belong to the set A on any subsets of S of positive measure. The system is in equilibrium when the potential (28) reaches a stationary point. The state equations are obtained by imposing:

$$0 \in \hat{c}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0), \tag{29}$$

where the symbol $\hat{c}$ indicates the subdifferential operators. Let $(\delta\mathbf{u}^{(1)}, \delta\mathbf{u}^{(2)}, \delta\mathbf{s}, \delta\mathbf{r}, \delta\gamma)$ be any admissible variation of the variables in the total potential energy, the condition (29) can be written in explicit form as:

$$0 \in \hat{c}_{\mathbf{u}^{(1)}}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0) \Leftrightarrow 0 = a^{(1)}(\mathbf{u}^{(1)0}, \delta\mathbf{u}^{(1)}) - f^{(1)}(\delta\mathbf{u}^{(1)}) + \int_S \mathbf{r}^0 \cdot \delta\bar{\mathbf{u}}^{(1)}\, \mathrm{d}s \tag{30a}$$

$$0 \in \hat{c}_{\mathbf{u}^{(2)}}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0) \Leftrightarrow 0 = a^{(2)}(\mathbf{u}^{(2)0}, \delta\mathbf{u}^{(2)}) - f^{(2)}(\delta\mathbf{u}^{(2)}) - \int_S \mathbf{r}^0 \cdot \delta\bar{\mathbf{u}}^{(2)}\, \mathrm{d}s \tag{30b}$$

$$0 \in \hat{c}_{\mathbf{s}}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0) \Leftrightarrow 0 = \int_S (\mathbf{r}^0 + \mathbf{t}^0) \cdot \delta\mathbf{s}\, \mathrm{d}s \tag{30c}$$

$$0 \in \hat{c}_{\mathbf{r}}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0) \Leftrightarrow 0 = \int_S [\mathbf{s}^0 - (\bar{\mathbf{u}}^{(2)0}) - \bar{\mathbf{u}}^{(1)0}) \cdot \delta\mathbf{r}]\, \mathrm{d}s \tag{30d}$$

$$0 \in \hat{c}_{\gamma}\pi(\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0, \gamma^0) \Leftrightarrow 0 = \int_S (\omega - Y^0 + q^0)\delta\gamma\, \mathrm{d}s, \tag{30e}$$

where, at each point of the surface S, $q^0 \in \hat{c}I_A(\gamma^0)$ and, according to eqns (15), $t^0 \in \partial_{\mathbf{s}} I_K(\mathbf{s}^0, \gamma^0)$ and $-Y^0 \in \hat{c}_{\gamma} I_K(\mathbf{s}^0, \gamma^0)$. Because of eqns (30c) the mechanical meaning of the Lagrange multiplier $\mathbf{r}$ becomes clear. It represents the opposite of the stress at the interface, hence it is the force per unit of area applied on the plate V_2 by means of the contact-adhesion with the plate V_1. On the other hand, it should be noted that the physical meaning of the quantity q^0 remains obscure. In fact, while it has a strong mathematical justification, its mechanical sense is not evident. It can be interpreted as the reactive density energy which ensures that the constraint $\gamma_1 \leqslant \gamma \leqslant 1$ is not violated.

It is interesting now to compute the subgradients $\partial I_K(\mathbf{s}, \gamma)$ and $\partial I_A(\gamma)$ at a typical point P of the surface S. When at the point P the initial damage parameter is $\gamma_1 < 1$, the following cases are possible:

$$\gamma = \gamma_1 \qquad \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \mathbf{R}^2 \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \{0\} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R}^- \tag{31a}$$

$$\gamma_1 < \gamma < 1 \qquad \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \mathbf{R}^2 \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \{0\} \quad \hat{c}_{\gamma} I_A(\gamma) = \{0\} \tag{31b}$$

$$\gamma = 1 \qquad s_n > 0 \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \{0\} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \mathbf{R} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R}^+ \tag{31c}$$

$$s_n = 0\mathbf{s}^{\mathrm{t}} \neq \mathbf{0} \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \mathbf{R} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R}^+ \tag{31d}$$

$$s_n = 0\mathbf{s}^{\mathrm{t}} = \mathbf{0} \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \{0\} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R}^+. \tag{31e}$$

On the contrary, when $\gamma_1 = 1$ at the point P, only the cases with $\gamma = 1$ are possible:

$$\gamma = 1 \qquad s_n > 0 \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \{0\} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \mathbf{R} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R} \tag{32a}$$

$$s_n = 0\mathbf{s}^{\mathrm{t}} \neq \mathbf{0} \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \mathbf{R} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R} \tag{32b}$$

$$s_n = 0\mathbf{s}^{\mathrm{t}} = \mathbf{0} \Rightarrow \hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma) = \mathbf{R} \times \{(0,0)\} \quad \hat{c}_{\gamma} I_K(\mathbf{s}, \gamma) = \mathbf{R} \quad \hat{c}_{\gamma} I_A(\gamma) = \mathbf{R}. \tag{32c}$$

The mechanical meaning of $\hat{c}_{\mathbf{s}} I_K(\mathbf{s}, \gamma)$, subset of $\mathbf{R}^3$, can be understood in the two possible cases: either there is adhesion ($\gamma < 1$) and then the interface stress vector $\mathbf{t} = t_n\mathbf{n} + \mathbf{t}^{\mathrm{t}}$ can

have any direction ($\mathbf{t} \in \mathbf{R}^3$), or there is not adhesion ($\gamma = 1$) then the interface stress is due only to the frictionless unilateral contact ($t_n \leqslant 0$, $\mathbf{t}^t = 0$).

A very interesting consequence of the governing equation (30e) can be carried out. Let eqn (30e) be written in its local form:

$$\omega - Y^0 + q^0 = 0. \tag{33}$$

When $\gamma_i < 1$, by taking into account eqns (31), eqn (33) excludes the possibilities of solutions of the form $\gamma_i < \gamma < 1$ or $\gamma = 1$ with $\mathbf{s} = \mathbf{0}$. In fact, if $\gamma_i < \gamma < 1$, eqn (31b) imposes $Y = q = 0$ and then eqn (33) cannot be satisfied because ω is a fixed positive quantity representing a material property of the interface; furthermore, if $\gamma = 1$ with $\mathbf{s} = \mathbf{0}$ eqn (31e) implies $Y = 0$ and $q \geqslant 0$, and those values are incompatible values for the eqn (33). Hence, when the initial value of the damage function at a typical point $\mathbf{P}$ of the interface S is $\gamma_i < 1$, the solution at this point for the adhesion problem is restricted to the cases:

$$\gamma_i < 1 \Rightarrow \begin{matrix} \gamma = \gamma_i & \mathbf{s} = \mathbf{0} \\ \gamma = 1 & \mathbf{s} \neq \mathbf{0} \end{matrix} \tag{34}$$

When $\gamma_i = 1$ admissible solutions are:

$$\gamma_i = 1 \Rightarrow \gamma = 1 \quad \begin{matrix} \mathbf{s} = \mathbf{0} \\ \mathbf{s} \neq \mathbf{0} \end{matrix} \tag{35}$$

In fact, eqns (32) make the solution (35) consistent with the local form of the governing equation (33). From a mechanical point of view, when in a typical point of the interface there is initially adhesion, i.e. $\gamma_i < 1$, then after the application of the external load, the solution can be adhesion without any variation of the damage parameter, or complete delamination with full damage and non-zero relative displacement of the surfaces. On the contrary, when the external load is applied in the typical point of the interface where initial complete damage exists, i.e. $\gamma_i = 1$, then the relative displacement solution may be different or equal to zero. This behavior seems to be realistic. In fact, if initially there is adhesion and a load is applied which tries to separate the two surfaces, then it can succeed or not; hence, if it succeeds in separating, separation occurs and adhesion is lost (the damage is complete). On the contrary if the load is not sufficient to separate the two surfaces, according to the present model, the damage parameter does not change. It can be deduced that only a brutal damage behavior is possible for the proposed model, like in elastic fracture mechanics.

6. TOPOLOGICAL REMARKS

A very important mathematical point is the choice of the topology adopted to compute the distance between two possible states $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$, such that it has a consistent physical meaning. In the paper by Frémond (1987) the fundamental difference from a mechanical point of view between the two situations characterized by $\gamma' = 1$ and $\gamma'' = 1 - \varepsilon$, with $0 < \varepsilon \ll 1$, on a finite area is emphasized. In fact, in the first case delamination occurs while in the second it is excluded. Nevertheless, the distance between the two situations according to the classical L^2 norm can be as small as possible since $\varepsilon \to 0$. As a consequence, the L^2 topology is not able to account for the great mechanical difference between the two situations. Hence, Frémond introduced a very special topology using a definition of distance involving the difference of the sets in adhesion for the two states. In this section it is proved that, because of the relations (34) and (35) between $\mathbf{s}$ and γ, the situation $\gamma'' = 1 - \varepsilon$, with $0 < \varepsilon \ll 1$, is excluded and hence the L^2 norm becomes satisfactory.

It can be supposed that at the beginning of any external load application, the damage function on S assumes the value $\gamma_i = 0$ on S_a where there is adhesion, and $\gamma_i = 1$ on S_d

where there is the initial defect. Hence, at beginning of the load history, partial adhesion is excluded. Then, eqns (34) assume the particular form:

$$\gamma_i = 0 \Rightarrow \begin{matrix} \gamma = 0 & \mathbf{s} = \mathbf{0} \\ \gamma = 1 & \mathbf{s} \neq \mathbf{0} \end{matrix} \tag{36}$$

As a consequence of eqns (36) the partial adhesion eventuality is impossible in every point of S during any load history.

Let $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$ be two possible solutions for the adhesion problem, i.e. $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$ satisfy the compatibility conditions (36) at each point of the subset $S_a \subset S$ where $\gamma_i = 0$, and condition (35) at each point of the subset $S_d \subset S$ where $\gamma_i = 1$. The following subsets of S are introduced:

$$S'_d = \{(x_1, x_2) : \gamma'(x_1, x_2) = 1\} \tag{37a}$$

$$S'_a = \{(x_1, x_2) : \gamma'(x_1, x_2) = 0\} \tag{37b}$$

$$S''_d = \{(x_1, x_2) : \gamma''(x_1, x_2) = 1\} \tag{37c}$$

$$S''_a = \{(x_1, x_2) : \gamma''(x_1, x_2) = 0\}. \tag{37d}$$

Obviously, $S_d \subset S'_d$, $S_d \subset S''_d$, $S'_a \subset S_a$ and $S''_a \subset S_a$. Let Δ denotes the distance between $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$ in the $L^2(S)$ norm. It means that:

$$\Delta = \|(\mathbf{s}' - \mathbf{s}'', \gamma' - \gamma'')\|_{L^2}^2 = \|\mathbf{s}' - \mathbf{s}''\|_{L^2}^2 + \|\gamma' - \gamma''\|_{L^2}^2 = \int_S (\mathbf{s}' - \mathbf{s}'')^2 \, ds + \int_S (\gamma' - \gamma'')^2 \, ds. \tag{38}$$

Four different regions can be distinguished in S:

$$S_1 = S'_d \cap S''_d \quad S_2 = S'_a \cap S''_a \quad S_3 = S'_d \cap S''_a \quad S_4 = S'_a \cap S''_d. \tag{39}$$

The value of the distance Δ defined by means of eqn (38) can be computed as the sum of the distances Δ_i, with $i = 1, 2, 3, 4$, evaluated on the four regions introduced by eqns (39). Thus:

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \tag{40}$$

where

$$\text{on } S_1 \quad \gamma' = 1 \quad \gamma'' = 1 \qquad \Delta_1 = \int_{S_1} (\mathbf{s}' - \mathbf{s}'')^2 \, ds \tag{41a}$$

$$\text{on } S_2 \quad \gamma' = 0 \quad \gamma'' = 0 \quad \mathbf{s}' = \mathbf{0} \quad \mathbf{s}'' = \mathbf{0} \qquad \Delta_2 = 0 \tag{41b}$$

$$\text{on } S_3 \quad \gamma' = 1 \quad \gamma'' = 0 \quad \mathbf{s}'' = \mathbf{0} \qquad \Delta_3 = \int_{S_3} (\mathbf{s}')^2 \, ds + \mu(S_3) \tag{41c}$$

$$\text{on } S_4 \quad \gamma' = 0 \quad \gamma'' = 1 \quad \mathbf{s}' = \mathbf{0} \quad \mathbf{s}'' \qquad \Delta_4 = \int_{S_4} (\mathbf{s}'')^2 \, ds + \mu(S_4) \tag{41d}$$

with $\mu(S_i)$ the measure of the subset S_i. Note that in eqns (41) when the vectors $\mathbf{s}'$ and $\mathbf{s}''$ are not specified, it means that they can assume any value. As a consequence of the relationships (39)–(41), the couples of functions $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$ are defined close to each other in the L^2 norm, when:

$$\text{on } S_1 \quad \Delta_1 = \|(\mathbf{s}' - \mathbf{s}'')^{-}\|_{L^2}^2 \leqslant \theta^2 \tag{42a}$$

$$\text{on } S_2 \quad \Delta_2 = 0 \tag{42b}$$

$$\text{on } S_3 \quad \Delta_3 = \|\mathbf{s}'\|_{L^2}^2 + \mu(S_3) \leqslant \theta^2 \tag{42c}$$

$$\text{on } S_4 \quad \Delta_4 = \|\mathbf{s}''\|_{L^2}^2 + \mu(S_4) \leqslant \theta^2, \tag{42d}$$

where $\theta \ll 1$. In particular, it can be deduced from eqns (42c,d) that two compatible state functions are close in the L^2 sense, only if:

$$\mu(S_3) \leqslant \theta^2 \quad \mu(S_4) \leqslant \theta^2 \tag{43a}$$

or, equivalently:

$$\mu(S_a' \cup S_a'' - S_a' \cap S_a'') \leqslant 2\theta^2. \tag{43b}$$

In other words, the functions $(\mathbf{s}', \gamma')$ and $(\mathbf{s}'', \gamma'')$ are close when S_d' and S_a' do not differ substantially from S_d'' and S_a'', respectively.

Finally, the topology induced by the L^2 norm appears to be satisfactory for the present problem. In fact two situations are close in the L^2 norm only if equation (43b) holds true, which means that the sets S_a' and S_a'' are close in the Lebesgue measure sense. When S_a' and S_a'' are regular sets, condition (43b) implies that they are close in the Hausdorff measure sense or that the lines of delaminations separating the bonded and the unbonded parts for the two considered cases, are close to each other (Chenais, 1976; Rodrigues, 1991). The present conclusion is not in contradiction with that proposed in (Frémond, 1987), where the L^2 induced topology appears unsatisfactory, since the Frémond's model is not constrained to satisfy the conditions (36) or (35).

7. THE EQUIVALENCE WITH THE GRIFFITH CRITERION

A very interesting feature of the present model is the possibility to recover the fracture mechanics approach (Maugis, 1987). The on/off character of the model allows one to suppose a close connection with fracture mechanics. The aim of this section is to give a mathematical proof that this supposition holds true.

To this end, let the solution state be decomposed as $\Sigma^0 = \{\sigma^0, \gamma^0\}$, with $\sigma^0 = \{\mathbf{u}^{(1)0}, \mathbf{u}^{(2)0}, \mathbf{s}^0, \mathbf{r}^0\}$. The state Σ^0 is a stationary point for the total potential energy $\pi(\Sigma)$ defined by eqn (28), which can be rewritten in the equivalent form:

$$\pi(\Sigma) = \pi(\sigma, \gamma) = E(\Sigma) - \int_S (1-\gamma)\omega \, \mathrm{d}s + \int_S I_A(\gamma) \, \mathrm{d}s, \tag{44}$$

where $E(\Sigma)$ represents the elastic energy of the two connected plates:

$$E(\Sigma) = \sum_{i=1}^{2} \left[\tfrac{1}{2} a^{(i)}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)}) - f^{(i)}(\mathbf{u}^{(i)})\right] + \int_S \mathbf{l}_K(\mathbf{s}, \gamma) \, \mathrm{d}s + \int_S \mathbf{r} \cdot [\mathbf{s} - (\bar{\mathbf{u}}^{(2)} - \bar{\mathbf{u}}^{(1)})] \, \mathrm{d}s. \tag{45}$$

Now, for a given damage function γ' the stationary condition for the total potential energy (44) leads to a classical linear elastic problem. In this case the solution state $\sigma' = \sigma(\gamma')$ satisfies only eqns (30a–d) and, moreover, it is unique. As a consequence, a one-to-one map between γ' and σ' is defined. The total potential energy associated to the state (σ', γ') is reduced to:

$$\hat{\pi}(\gamma') = \hat{E}(\gamma') - \int_S (1-\gamma')\omega \, ds + \int_S I_A(\gamma') \, ds \tag{46}$$

with

$$\hat{E}(\gamma') = \sum_{i=1}^{2} \left[\tfrac{1}{2} a^{(i)}(\mathbf{u}^{(i)}(\gamma'), \mathbf{u}^{(i)}(\gamma')) - f^{(i)}(\mathbf{u}^{(i)}(\gamma'))\right]. \tag{47}$$

Note that the elastic energy is differentiable with respect to $\mathbf{u}^{(*)}$. Thus, if $\mathbf{u}^{(*)}(\gamma)$ is differentiable then the functional $\hat{E}$ is also differentiable with respect to γ. If Σ^0 is a stationary point for π then γ^0 must be a stationary point for $\hat{\pi}$. Let $\delta\gamma$ be an admissible variation of the damage function, obtained as the difference $\delta\gamma = \gamma' - \gamma^0$ where γ' is a function close to the function γ^0, in the sense of L^2 as seen in the previous section. The function $\delta\gamma$ represents an admissible variation for the computation of the directional derivative of the potential $\hat{\pi}(\gamma')$. The stationary condition for the total potential energy $\hat{\pi}$ written in the form (46) leads to:

$$0 = \int_S (\omega + q^0)\delta\gamma \, ds + \delta_{\delta\gamma}\hat{E}(\gamma_0), \tag{48}$$

where $\delta_{\delta\gamma}\hat{E}(\gamma_0)$ represents the variation of the functional $\hat{E}$ evaluated at γ^0, along the direction $\delta\gamma$. By a simple comparison between eqn (30e) and its equivalent form (48), it can be deduced that:

$$-\int_S Y^0 \delta\gamma \, ds = \delta_{\delta\gamma}\hat{E}(\gamma_0). \tag{49}$$

It may be remarked that, at each point of the interface q, $\delta\gamma \leqslant 0$. In fact, when $\gamma = 0$ then $q \leqslant 0$ and $\delta\gamma \geqslant 0$, on the contrary when $\gamma = 1$ then $q \geqslant 0$ and $\delta\gamma \leqslant 0$. Hence, because of eqns (48) and (49), the following inequality holds:

$$\int_S (\omega - Y^0)\delta\gamma \, ds \geqslant 0. \tag{50}$$

Since γ^0 is supposed to be the damage function solution of the delamination problem, it belongs to the set of admissible damage functions satisfying conditions (35) or (36). Hence the function γ^0 assumes the values 0 or 1 on S. The support of the function γ^0 is denoted by S_{d^0}, and it represents the actual delaminated area. The function γ' is chosen such that it satisfies the conditions (35) or (36). The support of the function γ' is denoted by $S_{d'}$. Then, the set $\delta S_d = (S_{d^0} \cup S_{d'}) - (S_{d^0} \cap S_{d'})$ is the support of the function $\delta\gamma$. The variation $\delta\gamma$ can assume values 1 or -1 on δS_d and 0 outside.

In order to compare $-Y^0$ with the notion of energy release rate of fracture mechanics, two very special types of variations $\delta\gamma$ are considered: it is assumed on δS_d that $\delta\gamma$ takes either positive or negative values on its support. This can be obtained by setting $S_{d'} \supset S_{d^0}$ or $S_{d^0} \supset S_{d'}$. Firstly, the case $S_{d'} \supset S_{d^0}$ is considered. Note that the local value of the energy $Y^0(P)$ depends on the position of the point $P \in S$. In fact three cases can occur; (i) If P belongs to the interior of S_{d^0}, then the delaminated area cannot grow around the point P; (ii) If P is in the exterior of S_{d^0}, then let $S_{d'} = S_{d^0} + \mathcal{J}(P)$, where $\mathcal{J}(P)$ is a neighborhood of the point P; (iii) If, finally, P belongs to the boundary of S_{d^0}, then let $S_{d'} = S_{d^0} \cup \mathcal{J}(P)$. In the last two cases the local value of the energy of adhesion computed at the typical point P is given by:

$$Y^0(P) = \lim_{\mathscr{J}(P)\to P} \left\{ \frac{1}{\mu(S_d - S_{d^0})} \int_{\delta S_d} Y^0 \, dS \right\}, \tag{51}$$

with $\delta S_d = S_{d'} - S_{d^0}$. Since the correspondence between the damage function γ and its support S_d is bijective, the elastic energy of the laminate can be thought directly as a function of support of γ, i.e. with abuse of notation $\hat{E}(\gamma') = \hat{E}(S_d)$. As a consequence, because of eqn (51), it follows:

$$-Y^0(P) = \lim_{\mathscr{J}(P)\to P} \left\{ \frac{1}{\mu(S_d - S_{d^0})} [\hat{E}(S_d') - \hat{E}(S_{d^0})] \right\} = \left. \frac{d\hat{E}(S_{d^0})}{dS_d} \right|_P. \tag{52}$$

Analogously, for the case $S_{d'} \subset S_{d^0}$, three cases are possible, depending on the position of P. In fact, if P belongs to the interior of S_{d^0}, then $S_d = S_{d^0} - \mathscr{J}(P)$, if P belongs to the boundary of S_{d^0}, then $S_d = S_{d^0} \cap (S - \mathscr{J}(P))$, if, finally, P is outside S_{d^0}. In the first two cases eqn (52) is always obtained, where now $i\mu(S_d - S_{d0}) < 0$. Finally, the energy of adhesion can be regarded as the derivative of the elastic energy of the structure with respect to the variation of the surface of delamination, which is generally named in fracture mechanics as the energy release rate.

At this stage the meaning of inequality (50) appears clear. In fact, it can be deduced that:

$$P \notin S_{d^0} \quad \gtrless \quad -\left. \frac{d\hat{E}(S_{d^0})}{dS_d} \right|_P \leqslant \omega \tag{53a}$$

$$P \in \text{int}(S_{d^0}) \quad \gtrless \quad -\left. \frac{d\hat{E}(S_{d^0})}{dS_d} \right|_P \geqslant \omega \tag{53b}$$

$$P \in \partial S_{d^0} \quad \gtrless \quad -\left. \frac{d\hat{E}(S_{d^0})}{dS_d} \right|_P = \omega, \tag{53c}$$

where $\text{int}(S_{d^0})$ denotes the interior of the set S_{d^0}. Here the close connection with fracture mechanics theory clearly appears. When point P belongs to the delaminated area then the energy of adhesion Y^0 is greater than the Dupré's energy. When point P is in the adherent area then Y^0 is less than ω, and when point P is on the boundary of the delaminated area then Y^0 is equal to ω. Hence, the Dupré's energy is the Griffith limit energy release rate, and as a consequence can be evaluated by using standard experimental tests of fracture mechanics. If the crack zone is stable, according to the Griffith theory, the energy release rate is less than the limit energy release rate. On the contrary, if the energy release rate is greater than the limit energy release rate then the delaminated area grows (Destuynder, 1991).

8. THE REGULARIZED MODEL

The nondifferentiability of the functional π defined by eqn (28), and governing the delamination problem, could represent a problematic mathematical difficulty, both from a theoretical and computational point of view. The nondifferentiability of the total potential energy is due to the presence in the functional of the indicator functions I_K and I_A. In order to transform the nonsmooth problem into a new one governed by a differentiable functional, the regularization technique can be used. It consists in replacing in the original governing functional π the nondifferentiable terms with new differentiable ones, characterized by the fundamental property that they must tend to the original irregular funtionals when some parameters, called penalty or regularization parameters, approach to zero.

Several different regularizations can be chosen. Among all the possible regularizations, it is suitable to select the ones which have evident mechanical interpretations. In what follows, a very simple regularization is considered.

First regularization

The function I_K is regularized by considering the following differentiable function:

$$I_K^r = \frac{1}{2\eta_n}[(1-\gamma)(s_n^+)^2 + (s_n^-)^2] + \frac{1}{2\eta_t}(1-\gamma)\|\mathbf{s}_t\|^2, \tag{54}$$

where $(\cdot)^+$ and $(\cdot)^-$ indicate the positive and negative parts of $(\cdot)$, respectively, such that $(\cdot) = (\cdot)^+ - (\cdot)^-$, and $\|\cdot\|$ denotes the norm of a vector. The scalars $\eta_n > 0$ and $\eta_t > 0$ are penalty parameters. It can be seen that for $\eta_n \to 0$ and $\eta_t \to 0$ the function $I_K^r \to I_K$ from the exterior, i.e. I_K^r represents an exterior regularization of the perfect penalty function I_K.

Since the free energy density for the interface material is assumed to be equal to the indicator function of the set K (14), the regularization of I_K induces the regularization of the free energy density of the interface. Thus, eqns (15) defining the thermodynamical forces, i.e. the stress vector at interface and the energy of adhesion, assume the regularized form:

$$\mathbf{t} = \frac{1}{\eta_n}[(1-\gamma)s_n^+ - s_n^-]\mathbf{n} + (1-\gamma)\frac{1}{\eta_t}\mathbf{s}_t \tag{55a}$$

$$Y = \frac{1}{2\eta_n}(s_n^+)^2 + \frac{1}{2\eta_t}\|\mathbf{s}_t\|^2. \tag{55b}$$

According to the regularized equation (55a), the interface material is modeled by a uniform distribution of nonlinear elastic springs oriented in the normal and tangential directions to the surface S. The stiffness of the springs depends on the damage state. In the tangential direction the stiffness of the springs is $(1-\gamma)/\eta_t$. The springs in the normal direction have a nonlinear behavior that can be considered as bimodular. In fact, the stiffness in normal direction is $(1-\gamma)/\eta_n$ in traction and $1/\eta_n$ in compression.

By replacing the function I_K with its regularized form I_K^r into the functional π the new regularized total potential energy π^r is obtained. The stationary condition for the functional π^r leads to a first regularized form of eqns (30a–30e). The regularized equations are formally identical to the original ones, and define the solution $\Sigma^r = \{\sigma^r, \gamma^r\}$, with $\sigma^r = \{\mathbf{u}^{(1)r}, \mathbf{u}^{(2)r}, \mathbf{s}^r, \mathbf{r}^r\}$. For the regularized problem, the interface stress vector and the interface energy of adhesion present in eqns (30) are given now by the formulae (55). The local form of the regularized eqn (30e) is:

$$0 = \omega - \left[\frac{1}{2\eta_n}(s_n^+)^2 + \frac{1}{2\eta_t}\|\mathbf{s}_t^r\|^2\right] + q^r(\gamma^r). \tag{56}$$

It may be remarked that eqn (56), contrarily to eqn (33) obtained following the original nonsmooth approach, makes the partial damage case $0 < \gamma < 1$ possible. In fact, when $\omega = Y$ then $q = 0$. The condition $0 \in \partial I_A(\gamma)$ can be satisfied for any admissible value of the damage parameter, i.e. for $0 \leqslant \gamma \leqslant 1$. As a consequence, when $\omega = Y$ eqns (30e) and (55b) are not able to uniquely define the value of the damage parameter. In fact, in the typical point of the interface where the initial value of the damage parameter is zero, after the application of the external load, when the adhesion energy reaches the Dupré's energy, the present regularized delamination model imposes:

$$0 \leqslant \gamma \leqslant 1 \quad \dot{\gamma} \geqslant 0. \tag{57}$$

Hence, it is possible to determine neither the value of the partial damage nor the rate of the

damage. In order to eliminate such ambiguity, a further hypothesis should be introduced in the model. It may be supposed that when the Dupré's energy limit is reached for the adhesion energy, total damage is obtained. Thus, partial damage is excluded and a brutal damage behavior is considered as occurs for the nonsmooth case:

$$\gamma' = 0 \quad \text{when } Y' < 0$$
$$\gamma' = 1 \quad \text{when } Y' \geq 0. \tag{58}$$

It is very interesting to note that the quantity Y' obtained in eqn (55b) by simple differentiation, with respect to the damage parameter, of the function I_K' represents the regularization of the local energy release rate of fracture mechanics. In fact, let $\sigma' = \{\mathbf{u}^{(1)\prime}, \mathbf{u}^{(2)\prime}, \mathbf{s}', \mathbf{r}'\} = \sigma^{\mathrm{r}}(\gamma')$ be the solution of the regularized form of eqns (30a–30d) for a fixed damage state γ'. The regularized form of the elastic energy of the laminate (45), associated with the damage state γ', is given by:

$$\hat{E}^{\mathrm{r}}(\gamma') = \tfrac{1}{2}a^{(1)}(\mathbf{u}^{(1)\prime}, \mathbf{u}^{(1)\prime}) + \tfrac{1}{2}a^{(2)}(\mathbf{u}^{(2)\prime}, \mathbf{u}^{(2)\prime}) - f^{(1)}(\mathbf{u}^{(1)\prime}) - f^{(2)}(\mathbf{u}^{(2)\prime}) + I_K^{\mathrm{r}}(\mathbf{s}', \gamma'). \tag{59}$$

It can be proved that Y' can be obtained by derivation of the elastic energy, with respect to a variation of the adhesion surface. In fact, by substituting eqn (30c) into (30a,b), by making the sum of the obtained equations, and by taking into account the condition (30d), for the state σ'

$$a^{(1)}(\mathbf{u}^{(1)\prime}, \delta\mathbf{u}^{(1)}) + a^{(2)}(\mathbf{u}^{(2)\prime}, \delta\mathbf{u}^{(2)}) - f^{(1)}(\delta\mathbf{u}^{(1)}) - f^{(2)}(\delta\mathbf{u}^{(2)}) + \int_S \mathbf{t}' \cdot \delta\mathbf{s}\, \mathrm{d}s = 0. \tag{60}$$

holds true. Now, by taking into account the relation (55a), by setting $\delta\mathbf{u}^{(1)} = \mathbf{u}^{(1)\prime}$, $\delta\mathbf{u}^{(2)} = \mathbf{u}^{(2)\prime}$ and $\delta\mathbf{s} = \mathbf{s}'$, and recalling that:

$$[(1-\gamma')s_{\mathrm{t}}^{\mathrm{r}} + s_{\mathrm{t}}']s_{\mathrm{t}}' = (1-\gamma')(s_{\mathrm{t}}^{\mathrm{r}})^2 + (s_{\mathrm{t}}')^2 \tag{61}$$

eqn (60) becomes:

$$a^{(1)}(\mathbf{u}^{(1)\prime}, \mathbf{u}^{(1)\prime}) + a^{(2)}(\mathbf{u}^{(2)\prime}, \mathbf{u}^{(2)\prime}) - f^{(1)}(\mathbf{u}^{(1)\prime}) - f^{(2)}(\mathbf{u}^{(2)\prime}) - \int_S \left\{ \frac{1}{\eta_{\mathrm{n}}} [(1-\gamma')(s_{\mathrm{n}}^{\mathrm{r}})^2 + (s_{\mathrm{n}}')^2] + \frac{1}{\eta_{\mathrm{t}}}(1-\gamma')\|\mathbf{s}_{\mathrm{t}}'\|^2 \right\} \mathrm{d}s = 0. \tag{62}$$

A comparison of the definition formulae (59) and (54) with the deduced eqn (62) allows one to write:

$$\hat{E}^{\mathrm{r}}(\gamma') = -\tfrac{1}{2}f^{(1)}(\mathbf{u}^{(1)\mathrm{r}}(\gamma')) - \tfrac{1}{2}f^{(2)}(\mathbf{u}^{(2)\mathrm{r}}(\gamma')). \tag{63}$$

Of course, eqn (60) is satisfied for both states σ^{r} and σ'. Then, consider the variation $\delta\sigma = \sigma^{\mathrm{r}}$ when eqn (60) holds for the state σ', and the variation $\delta\sigma = \sigma'$ when eqn (60) holds for the state σ^{r}. In these cases, eqn (60) becomes, respectively:

$$f^{(1)}(\mathbf{u}^{(1)\mathrm{r}}) + f^{(2)}(\mathbf{u}^{(2)\mathrm{r}}) = a^{(1)}(\mathbf{u}^{(1)\prime}, \mathbf{u}^{(1)\mathrm{r}}) + a^{(2)}(\mathbf{u}^{(2)\prime}, \mathbf{u}^{(2)\mathrm{r}}) - \int_S \left\{ \frac{1}{\eta_{\mathrm{n}}} [(1-\gamma')s_{\mathrm{n}}^{\mathrm{r}} s_{\mathrm{n}}'] s^{\mathrm{r}} + \frac{1}{\eta_{\mathrm{t}}}(1-\gamma')\mathbf{s}_{\mathrm{t}}' \mathbf{s}_{\mathrm{t}}^{\mathrm{r}} \right\} \mathrm{d}s \tag{64}$$

$$f^{(1)}(\mathbf{u}^{(1)r})+f^{(2)}(\mathbf{u}^{(2)r}) = a^{(1)}(\mathbf{u}^{(1)r},\mathbf{u}^{(1)r})+a^{(2)}(\mathbf{u}^{(2)r},\mathbf{u}^{(2)r}) + \int_S \left\{ \frac{1}{\eta_n}[(1-\gamma^r)s_n^{r+}-s_n^{r-}]s_n' + \frac{1}{\eta_t}(1-\gamma^r)\mathbf{s}_t^r\cdot\mathbf{s}_t' \right\} ds. \quad (65)$$

The difference between the damage function γ' and the damage function γ^r, solution of the regularized delamination problem, represents an admissible variation for the damage function $\delta\gamma = \gamma' - \gamma^r$, useful for the computation of the derivative of the elastic energy with respect to the change of adhesion surface. The damage function γ' is chosen such that $\delta\gamma = \gamma' - \gamma^r$ assumes values of $+1$ or -1 on δS_d. In the following the classical notations are used and the case of point P belonging to the boundary of the delamination area is considered. As a consequence, because of eqns (63)–(65) the difference between the elastic energies of the laminate computed for the states σ' and σ^r can be written in the form:

$$\hat{E}^r(\gamma') - \hat{E}^r(\gamma^r) = -\left(\int_{S_d} \frac{1}{2\eta_n} s_n^{r-} s_n' \, ds - \int_{S_d} \frac{1}{2\eta_n} s_n'^{-} s_n^r \, ds + \int_{S_d} \frac{1}{2\eta_t} \mathbf{s}_t^r \cdot \mathbf{s}_t' \, ds - \int_{S_d} \frac{1}{2\eta_t} \mathbf{s}_t' \cdot \mathbf{s}_t^r \, ds + \int_S \frac{1}{2\eta_n} (s_n'^{-} s_n^r - s_n^{r-} s_n') \, ds \right), \quad (66)$$

where the classical notations are used for the definition of the subsets of S and the symmetry of the bilinear forms $a^{(1)}$ and $a^{(2)}$ is taken into account. In what follows, it is assumed that the state σ' depends continuously on γ', thus if γ' tends to γ^r then s' tends to s^r. Finally, the derivative of the elastic energy with respect to a variation of adhesion surface is given by:

$$\left.\frac{d\hat{E}(\gamma^r)}{dS_d}\right|_P = \lim_{\mathscr{I}(P)\to P} \left\{ \frac{1}{\mu(S_d' - S_d^0)} [\hat{E}(\gamma') - \hat{E}(\gamma^r)] \right\}$$

$$= -\lim_{\mathscr{I}(P)\to P} \left\{ \frac{1}{\mu(S_d' - S_d^0)} \left[\frac{1}{2\eta_n} \left(\int_{S_d} s_n^{r+} s_n' \, ds - \int_{S_d} s_n'^{+} s_n^r \, ds \right) \right] \right\}$$

$$-\lim_{\mathscr{I}(P)\to P} \left\{ \frac{1}{\mu(S_d' - S_d^0)} \left[\frac{1}{2\eta_t} \left(\int_{S_d} s_t^r \cdot \mathbf{s}_t' \, ds - \int_{S_d} \mathbf{s}_t' \cdot \mathbf{s}_t^r \, ds \right) \right] \right\}$$

$$-\lim_{\mathscr{I}(P)\to P} \left\{ \frac{1}{\mu(S_d' - S_d^0)} \frac{1}{2\eta_n} \int_S (s_n'^{-} s_n^r - s_n^{r-} s_n') \, ds \right\}. \quad (67)$$

It can be proved that the last limit in eqn (67) is zero. The mathematical proof is not straightforward and for sake of simplicity is not reported herein. A simple justification of this result can be obtained assuming that s' tends to s^r in the Lipschitzian manner with respect to the diameter ρ of the neighborhood $\mathscr{I}(P)$ of a point $P \in S$:

$$\mathbf{s}' - \mathbf{s}^r \leqslant c\rho \quad c \in \mathbf{R}^+. \quad (68)$$

In fact, the quantity in the last integral of eqn (67) is different from zero only when the product $s_n' s_n^r < 0$, and in this case the following chain of inequalities holds:

$$|s_n'^{-} s_n^r - s_n^{r-} s_n'| \leqslant |s_n' s_n^r| \leqslant (s_n' - s_n^r)^2, \quad (69)$$

where $|\cdot|$ indicates the absolute value of a number. By taking into account the hypothesis (68) and recalling that $\mu(S_d' - S_d^0) = \mu(\mathscr{I}(P))$, the relation (69) implies:

$$\frac{1}{\mu(S_d^t - S_d^0)} \int_{\mathscr{S}} |s_n^t \, s_n^t - s_n^t \, s_n^t| \, ds \leq \frac{(c\rho)^2}{\mu(\mathscr{J}(P))} \mu(\mathscr{S}), \tag{70}$$

where $\mathscr{S}$ is the subset of S in which $s_n^t s_n^t < 0$. Then, since the neighborhood of P varies as the square of its diameter when it tends to 0, it holds that:

$$\lim_{\mathscr{J}(P) \to P} \frac{(c\rho)^2}{\mu(\mathscr{J}(P))} = c_0 \quad c_0 \in \mathbf{R}^+. \tag{71}$$

Furthermore, the measure of the subset $\mathscr{S}$ where $s_n^t s_n^t < 0$ tends to zero when s^t tends to s, then:

$$\lim_{\mathscr{J}(P) \to P} \mu(\mathscr{S}) = 0. \tag{72}$$

Equations (71) and (72) allow one to deduce that the first term of the inequality (70) goes to zero when $\mathscr{J}(P)$ tends to P, and hence the last limit in eqn (67) is zero.

According to the value of the variation $\delta\gamma$ on δS_d, it may occur that $S_a \subset S_a^t$ or $S_a^t \subset S_a$. In fact, when $\delta\gamma = 1$ then $S_a^t \subset S_a$ and thus $S_a = S_a^t \cup \mathscr{J}$, on the contrary, when $\delta\gamma = -1$ then $S_a \subset S_a^t$ and thus $S_a^t = S_a \cup \mathscr{J}$. In both cases eqn (67) can be written in the equivalent form:

$$\left.\frac{d\hat{E}(\gamma^t)}{dS_d}\right|_P = -\lim_{\mathscr{J}(P) \to P} \frac{1}{\mu(S_d^t - S_d^0)} \left[\frac{1}{2\eta_n} \int_{\mathscr{J}S_d} s_n^t \, s_n^t \, ds + \frac{1}{2\eta_t} \int_{\mathscr{J}S_d} \mathbf{s}_t^t \cdot \mathbf{s}_t^t \, ds \right]$$

$$= -\left[\frac{1}{2\eta_n} (s_n^{t+})^2 + \frac{1}{2\eta_t} |\mathbf{s}_t^t|^2 \right]. \tag{73}$$

By a direct comparison between the obtained explicit formula of the derivative of the elastic energy (73) with the formula (55b), it can be deduced that even for the regularized problem, the energy of adhesion per unit area can be regarded as the energy release rate:

$$Y^t\big|_P = -\left.\frac{d\hat{E}^t(\gamma^t)}{dS_d}\right|_P. \tag{74}$$

Second regularization

A further simplification of the delamination problem can be obtained by a regularization of the function I_A. To this end, consider the differentiable function I_A^ε defined as:

$$I_A^\varepsilon(\gamma) = -\varepsilon(|1-2\gamma| + \ln|1-|1-2\gamma|). \tag{75}$$

It represents an internal regularization of the nonsmooth function I_A for $\gamma \in]0,1[$. The scalar $\varepsilon > 0$ is the penalty parameter. In fact, in the limit as $\varepsilon \to 0$ the function $I_A^\varepsilon \to I_A$ from the interior.

As a consequence of the further regularization, a new regularized total potential energy π^R is introduced by changing, in the functional π^t, I_A with I_A^ε. The stationary condition for π^R leads to eqns (30a–30e), where in eqn (30e) the quantity q^t is now the derivative of I_A^ε with respect to γ.

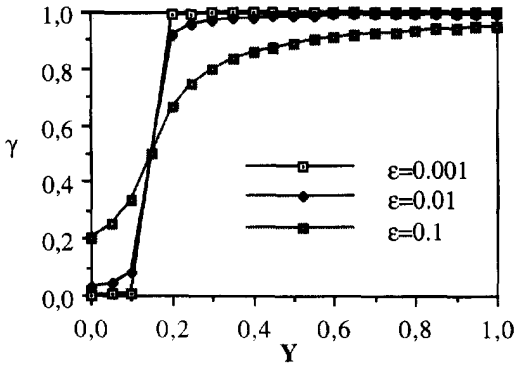

Fig. 2. Graph of the function $\gamma(Y)$.

$$q^{\varepsilon}(\gamma) = \varepsilon \frac{2\gamma - 1}{1 - |1 - 2\gamma|}. \tag{76}$$

The function $q^{\varepsilon}(\gamma)$ is not defined for $\gamma = 1$ and $\gamma = 0$. It is easy to verify that:

$$\lim_{\gamma \to 0^+} q^{\varepsilon}(\gamma) = -\infty \qquad \lim_{\gamma \to 1^-} q^{\varepsilon}(\gamma) = +\infty. \tag{77}$$

When the regularization parameter ε tends to 0, the graph of the function $q^{\varepsilon}(\gamma)$ tends to the graph of the subdifferential ∂I_A.

It should emphasized that the function $q^{\varepsilon}(\gamma)$ is bijective from $]0, 1[$ to $\mathbf{R}$. As a consequence, the restriction of the function $q^{\varepsilon}(\gamma)$ to $\gamma \in]0, 1[$ is invertible. Hence, taking into account eqn (76), eqn (56) can be inverted and then used to define the damage parameter corresponding to a certain relative displacement $\mathbf{s}$, as:

$$\gamma = \frac{\varepsilon - \left\{\omega - \left[\frac{1}{2\eta_n}(s_n^-)^2 + \frac{1}{2\eta_t}|\mathbf{s}_t|^2\right]\right\} + \left|\omega - \left[\frac{1}{2\eta_n}(s_n^+)^2 + \frac{1}{2\eta_t}|\mathbf{s}_t|^2\right]\right|}{2\left(\varepsilon + \left|\omega - \left[\frac{1}{2\eta_n}(s_n^+)^2 + \frac{1}{2\eta_t}|\mathbf{s}_t|^2\right]\right|\right)} \tag{78}$$

under the condition $\gamma \in]0, 1[$. The plot of the damage function $\gamma(Y)$, as function of the adhesion energy, obtained by combining eqn (78) with eqn (55b), is given in Fig. 2 for several values of the regularization parameter ε, when $\omega = 0.15$.

Now, let Σ^{ε} be the solution of the full regularized delamination problem governed by eqns (30a–30d), (55) and (78). When the regularization parameters ε, η_n and η_t tend to 0 the sequence of solutions $\{\Sigma^{\varepsilon}\}$ is expected to tend to the Σ^0 solution of the original problem (30). The convergence of the solution of the regularized problem to the solution of the nonsmooth one is treated in Point and Sacco (1995a).

It should be emphasized that the second regularization proposed leads to the simplest model which can account for a gradual damage behavior of the interface. In fact, the original non-smooth delamination approach described in Section 4 can generate many other regularized interface models just by choosing different expressions for I_K^{ε} and I_A^{ε}. Of course more complex models can be defined when I_K^{ε} and I_A^{ε} depend on several parameters which are to be set to fit with experimental results.

9. THE NUMERICAL PROCEDURE

In order to obtain a problem with a lower number of explicit unknowns, equations (30a–30d) and (55) governing the regularized problem of the delamination are reduced to:

$$0 = a^{(1)}(\mathbf{u}^{(1)r}, \delta\mathbf{u}^{(1)}) - f^{(1)}(\delta\mathbf{u}^{(1)}) - \int_S \left\{ \frac{1}{\eta_n} [(1-\gamma^r)s_n^r - s_n^c]\mathbf{n} + \frac{1}{\eta_t}(1-\gamma^r)\mathbf{s}_t^r \right\} \cdot \delta\bar{\mathbf{u}}^{(1)} \, ds \tag{79a}$$

$$0 = a^{(2)}(\mathbf{u}^{(2)r}, \delta\mathbf{u}^{(2)}) - f^{(2)}(\delta\mathbf{u}^{(2)}) + \int_S \left\{ \frac{1}{\eta_n} [(1-\gamma^r)s_n^r - s_n^c]\mathbf{n} + \frac{1}{\eta_t}(1-\gamma^r)\mathbf{s}_t^r \right\} \cdot \delta\bar{\mathbf{u}}^{(2)} \, ds, \tag{79b}$$

where $\mathbf{s}^r = \bar{\mathbf{u}}^{(2)r} - \bar{\mathbf{u}}^{(1)r}$ is implicitly accounted. Of course the problem is completed by means of eqn (78). In this manner, only the displacement and the damage fields appear in the governing equations.

The numerical procedure developed in the following is based on the finite element formulation. The structure is partitioned into finite elements. The discretization is realized for the plate Ω_1, for the plate Ω_2 and for the interface S. Thus, two different types of elements are obtained: the plate and the interface elements. The displacement parameters are expressed by the following relations:

$$\mathbf{u}^{(\alpha)r} = \sum_{i=1}^{n} \mathbf{u}_i^{(\alpha)r} \Psi^i \qquad \varphi^{(\alpha)r} = \sum_{i=1}^{n} \varphi_i^{(\alpha)r} \Psi^i. \tag{80}$$

with n number of nodes of the mesh and Ψ^i Lagrangian interpolation functions. By substituting the formulae (80) into eqns (79) the following algebraic problem is reached:

$$\begin{bmatrix} \mathbf{K}^{\Omega_1} + \mathbf{K}^S(\mathbf{U}^2 - \mathbf{U}^1, \gamma) & -\mathbf{K}^S(\mathbf{U}^2 - \mathbf{U}^1, \gamma) \\ -\mathbf{K}^S(\mathbf{U}^2 - \mathbf{U}^1, \gamma) & \mathbf{K}^{\Omega_2} + \mathbf{K}^S(\mathbf{U}^2 - \mathbf{U}^1, \gamma) \end{bmatrix} \begin{Bmatrix} \mathbf{U}^1 \\ \mathbf{U}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix}, \tag{81}$$

where $\mathbf{U}^i$ collects the displacement parameters of the plate Ω_i, $\mathbf{F}_i$ is the vector of the nodal forces acting on the plate Ω_i, $\mathbf{K}^{\Omega}$ is the stiffness matrix of the plate Ω_i, and $\mathbf{K}^S(\mathbf{U}^2 - \mathbf{U}^1, \gamma)$ is the secant stiffness matrix of the interface. Equation (81) can be written in the simpler form:

$$\mathbf{R}(\mathbf{U}, \gamma) = \mathbf{K}(\mathbf{U}, \gamma)\mathbf{U} - \mathbf{F} = 0, \tag{82}$$

where $\mathbf{U} = \{\mathbf{U}^2, \mathbf{U}^1\}$, $\mathbf{F} = \{\mathbf{F}^2, \mathbf{F}^1\}$ and $\mathbf{K}(\mathbf{U}, \gamma)$ is the secant stiffness matrix of the laminate. Equation (82) defines the equilibium condition at the generic instant t. In order to determine the displacement and damage evolution for the structure subjected to a certain load path, a numerical time integration is needed. Because of the softening interface constitutive law, a kinematically driven integration is recommended. Thus, let $\mathbf{U}^{[n]}$ and $\gamma^{[n]}$ be the displacement vector and the damage parameter at the time $t = t_n$. The numerical integration algorithm adopted belongs to the so-called predictor-corrector family and, at the (i)-th iteration for the $[n+1]$-th time step, consists of the following steps:

(i) define the tangent stiffness matrix $\mathbf{K}_t(\mathbf{U}_{(i)}^{[n+1]}, \gamma_{(i)}^{[n+1]}) = (\partial/\partial\mathbf{U})[\mathbf{K}(\mathbf{U}_{(i)}^{[n+1]}, \gamma_{(i)}^{[n+1]})]$ (83a)

(ii) compute the unbalanced force vector $\mathbf{G} = \mathbf{F} - [\mathbf{K}(\mathbf{U}_{(i)}^{[n+1]}, \gamma_{(i)}^{[n+1]})]\mathbf{U}_{(i)}^{[n+1]}$ (83b)

(iii) solve the linear problem $\Delta\mathbf{U}_{(i+1)}^{[n+1]} = [\mathbf{K}_t(\mathbf{U}_{(i)}^{[n+1]}, \gamma_{(i)}^{[n+1]})]^{-1}\mathbf{G}$ (83c)

(iv) update the displacement vector $\mathbf{U}_{(i+1)}^{[n+1]} = \mathbf{U}_{(i)}^{[n+1]} + \Delta\mathbf{U}_{(i+1)}^{[n+1]}$ (83d)

(v) determine the new damage function $\gamma_{(i+1)}^{[n+1]}$ by means of the formula (78).

(vi) go to the step (i) until the norm of the unbalanced force vector $\|\mathbf{G}\|$ is less than a prefixed tolerance.

Note that at beginning of each time step increment it is taken $\mathbf{U}_{(0)}^{[n+1]} = \mathbf{U}^{[n]}$, thus the first iteration of the $[n+1]$-th time step represents the elastic predictor, the second iteration can be regarded as the check of the consistency and then there is the damage corrector.

One of the major computational problems consists in a satisfactory determination of the stiffness matrix for the interface elements. In fact, the function to integrate in each finite element obtained as discretization of the interface S could be irregular enough, such that the classical Gauss integration can fail. This is due to the fact that each element can have a part in adhesion and another delaminated. This difficulty, occuring in many unilateral problems is generally solved by using a very fine mesh, in order to have in each element full adhesion or full delamination. This approach is not very effective. In fact, in this way the number of unknowns of the problem increases. In a recent paper (Barbero *et al.*, 1993) the Simpson's numerical integration method with a large number of integration points has been proposed to overcome this difficulty. The method appears very simple in the implementation and accurate in the determination of the stiffness of the interface elements even when a fine mesh is not adopted. Two integrations are then implemented for the interface elements: the Gauss and the Simpson routines. The first is used when the adhesion is total on the element, the second when the adhesion is partial on the element. When the element is completely delaminated the interface subroutine is skipped for that element.

10. NUMERICAL RESULTS

The non-smooth and the regularized models are adopted for the computations. The non-smooth model may be convenient for the analysis of the delamination of beams, since it allows us to obtain analytic solutions at least in simple cases. On the contrary, regularized models can be simply implemented in standard finite element codes, which allows one to approach more complex problems, e.g. delamination of plates. Furthermore, they allow the consideration of a finite deformability of the interface.

Hence, numerical results are carried out for beams using both the proposed finite element procedure with the regularized models, and the analytical approach developed in Point and Sacco (1995b) with the non-smooth and the first regularized models. Computations are carried out by neglecting the shear deformation in the thickness according to the Euler beam theory. Then, a plate problem is solved via the finite element method adopting the first regularized interface model.

The beam problem

Initially the problem of two beams in adhesion with each other is considered (Fig. 3). The two beams are supposed to be geometrically equal and realized with the same material. In what follows, E denotes the Young's modulus, $b \times h$ the dimensions of the cross-section of the beams, L the total length of the beams and L_2 the length of the initial defect. The following data are considered for the computations: $E = 130$ GPa, $b = 200$ mm, $h = 20$ mm, $L = 1000$ mm, $L_2 = 20$ mm. Both beams are clamped for $z = 0$ and beam 2 is subjected to the imposed transversal displacement $\delta = 10$ mm for $z = L$. In this first application delamination growth is not investigated, i.e. a linear problem is treated.

Analytical results are obtained using the brutal damage behavior for the first regularized model. Hence the problem is reduced to the analysis of two beams partially in contact with each other by means of vertical and horizontal distributed elastic springs with limited strength. Let k denote the stiffness of the vertical springs and c the stiffness of the horizontal springs. Note that k and c represent the penalty parameter b/η_n and b/η_t,

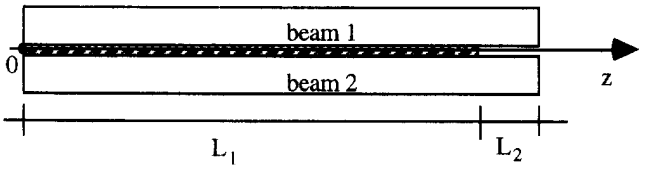

Fig. 3. Two beams partially connected by means of horizontal and vertical distributed springs.

Table 1. Analytical solution for the double cantiler beam problem

k (GPa)	Δv (mm)	Δw (mm)	F (kN)	γ (Nmm mm^{-1})
1.00e-3	2.19593	2.65166e-1	0.92242	2.42864
1.00e-2	4.61477e-1	2.89929e-1	1.03621	1.27495
1.00e-1	1.04769e-1	2.75091e-1	1.24683	2.44070
1.00	3.85067e-2	1.62347e-1	2.34380	7.33050
1.00e1	1.27387e-2	3.30997e2	3.75587	3.55036
1.00e2	3.09028e-3	4.79295e-3	4.11050	1.05180
1.00e3	7.50673e-4	8.58081e-4	4.15417	0.46583
1.00e4	1.95475e-4	2.04217e-4	4.15900	0.29531
1.00e5	5.43211e-5	5.78428e-5	4.15959	0.23118
1.00e6	1.58441e-5	1.76174e-5	4.15970	0.20311
1.00e7	4.77312e-6	5.50369e-6	4.15974	0.18964
FM solution	0.00000	0.00000	4.15976	0.17470

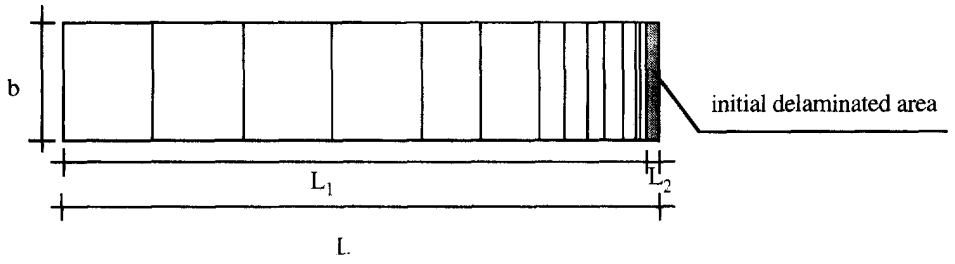

Fig. 4. Finite element discretization.

respectively. Several values for the stiffness of the vertical and horizontal springs are considered, with $c = k/2$.

The relative transversal and longitudinal displacements Δv and Δw for $z = L_1$, the reaction F of the support where the displacement δ is imposed, and the adhesion energy per unit of length $\gamma = Yb$ evaluated for $z = L_1$, obtained for several values of the stiffness k, are given in Table 1, where the "FM solution" represents the solution carried out adopting the non-smooth interface model, and which is in fact the fracture mechanics solution. It should be emphasized that the solutions carried out by means of the first regularized model converge toward the solution of the non-smooth problem and hence to the fracture mechanics solution, when the stiffness of the distributed springs tends to infinity. Then the problem is approached via finite element method. In the computations the second regularized model is adopted, with $j = \varepsilon b = 10^{-4}$ Nmm mm^{-1}. The mesh used for the computation is presented in Fig. 4. It consists of 14+14 beam elements with 13 special interface elements. For both the beam and the interface elements quadratic interpolation functions are used. Results are given in Table 2. The very good agreement between the analytical and the FEM results may be noted, by comparing Tables 1 and 2.

Next the nonlinear problem of the delamination is considered. To this end, the preliminary computation of the release energy rate via fracture mechanics has been developed for several values of the length L_1, and for several values of the transversal imposed displacement δ. The results obtained are plotted in Fig. 5. Several behaviors are possible, depending on the initial delamination defect and the Dupré's energy. In particular, for $\delta = 1$ mm, when a value of the Dupré's energy per unit of length less than $\bar{\omega} = \omega b = 0.068$ Nmm mm^{-1} is considered, complete separation of the two beams occurs if the length of

Table 2. Finite element solution for the double cantiler beam problem

k (GPa)	Δv (mm)	Δw (mm)	F (kN)	γ (Nmm mm^{-1})
1.00e2	3.10000e-3	4.81000e-3	4.11060	1.05890

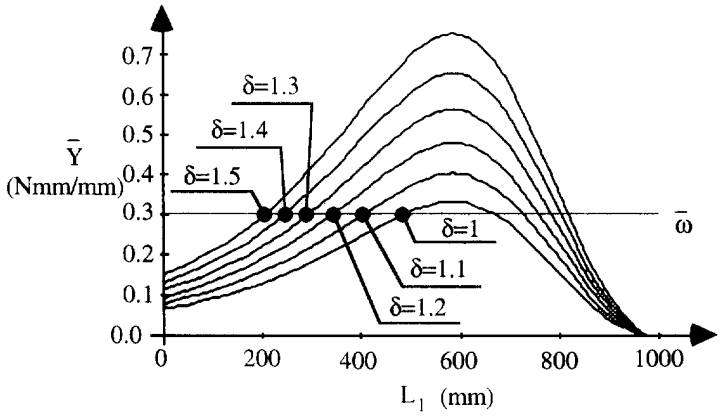

Fig. 5. Energy release rate vs. L_1 for several values of δ.

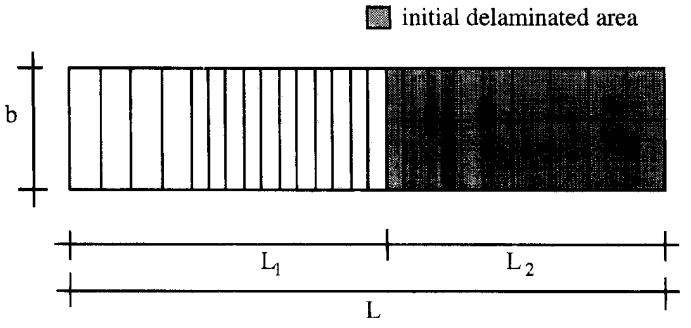

Fig. 6. Finite element mesh used to compute the delamination.

the initial defect is greater than 130 mm, as can be qualitatively deduced from Fig. 5. On the contrary, if the initial defect is smaller than 130 mm, delamination does not occur.

In order to investigate the delamination growth as a function of the imposed transversal displacement, the Dupré's energy per unit of length is taken $\bar{\omega} = 0.3$ Nmm mm^{-1}, and the big initial defect $L_2 = 465$ mm is considered. Results are obtained by using the analytical and the finite element approaches, with the first and the second regularized model, respectively. The parameters chosen for the application are: $k = 100$ GPa, $c = k/2$ and $j = \varepsilon b = 10^{-4}$ Nmm mm^{-1}. The mesh used for the FEM computations is presented in Fig. 6. The smallest elements are present close to the crack tip and in its neighborhood where delamination develops. The delamination length L_2 and the reaction F of the constraint, where the imposed transversal displacement is assumed, vs. the value of δ are plotted in Figs 7 and 8, respectively. These figures show the efficiency of the FEM developed for the determination of both the length of delamination and the displacement–force relationship.

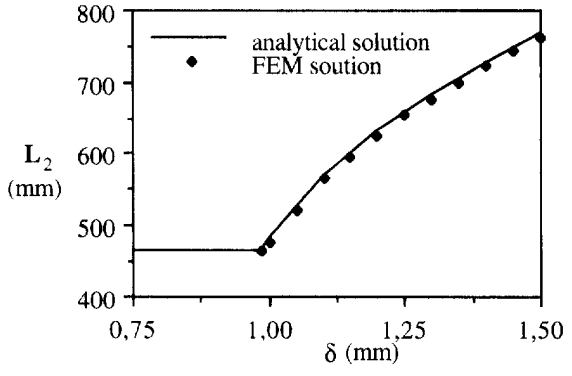

Fig. 7. Delamination length vs. imposed displacement.

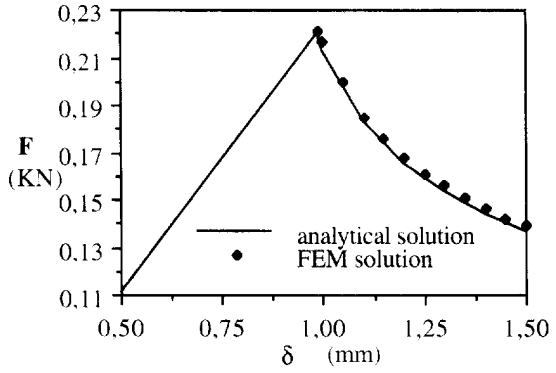

Fig. 8. Reaction vs. imposed displacement.

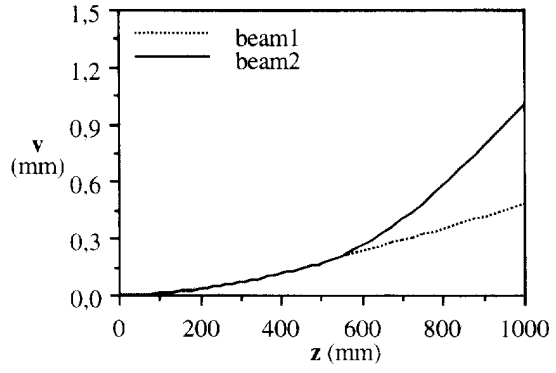

Fig. 9. Transversal displacements vs. z for $\delta = 1$ mm.

In fact, a very good agreement of the results obtained by means the analytical and the FEM approaches can be seen. In Figs 9 and 10 the transverse displacements v of the beams are plotted as a function of the coordinate z, for the cases $\delta = 1$ and 1.5 mm.

Finally, the effect of the regularization parameter $\varepsilon = a/b$ is investigated. Solutions are carried out via FEM for several values of ε, with the transversal imposed displacement $\delta = 1$ mm. In Fig. 11 the plot of the damage function γ vs. the z-axis, in the neighbourhood of the delamination line, is given. The diffusion of the damage for high values of the parameter ε and the convergence of the model to the brutal damage behavior when $\varepsilon \to 0$ should be noted.

The plate problem

The delamination problem of a composite laminate is treated. The analysis refers to a simply supported square plate with side $a = 500$ mm and thickness $h = 60$ mm. The plate

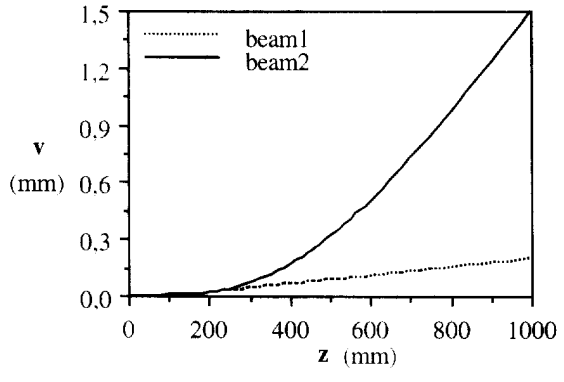

Fig. 10. Transversal displacements vs. z $\delta = 1.5$ mm.

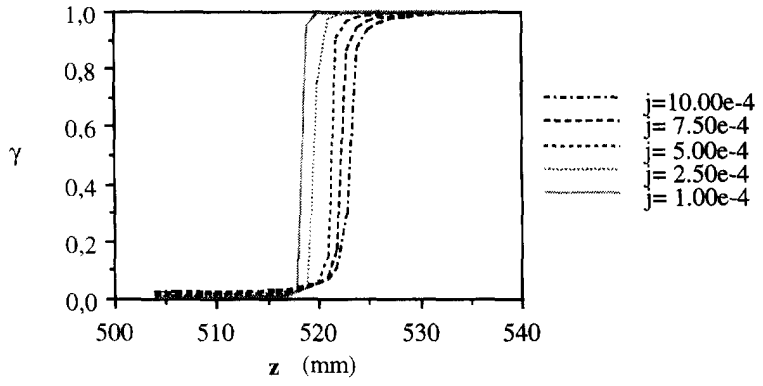

Fig. 11. Damage function vs. z, for several values of the internal regularization parameter.

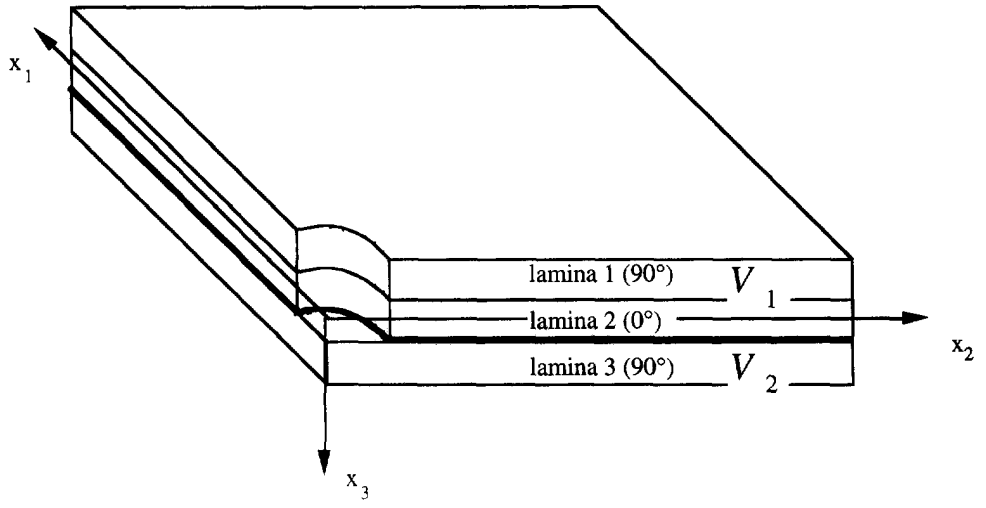

Fig. 12. A quarter of a cross-ply laminate with a hole in the first two laminae.

consists of three orthotropic laminae of "Fiberite T300/9762", which is a graphite–epoxy composite characterized by the following material properties (expressed in GPa): $E_L = 130$, $E_T = 9.65$, $G_{LT} = G_{L3} = 5.6$, $G_{T3} = 4.8$, $\nu_{LT} = 0.29$. In the center of the cross-ply (90°/0°/90°) laminate a circular hole with radius $\rho = 30$ mm is present only in the first two laminae of the plate. The continuity of the third lamina is warranted as shown in (Fig. 12), where only a quarter of the plate is represented. The Dupré's energy is taken $\omega = 0.15$ Nmm mm^{-2}. The first regularization parameters are: $\eta_n = 50$ GPa mm and $\eta_t = 2\eta_n$. The central point of the third lamina is subjected to a transversal imposed displacement δ. The delamination is supposed to develop between the second and the third lamina. Thus, the plate V_1 is composed of the first two laminae and V_2 of the third one. Because of the double symmetry of the structure only a quarter of the laminate is considered for the analysis. Hence, Ω_1 and Ω_2 are discretized in 24 and 25 plate elements, respectively, and S is discretized in 24 special interface elements.

Computations are developed for several values of the imposed transversal displacement δ. In Fig. 13 the separation lines between the parts of the interface S in adhesion and in delamination are presented. It should be noted that the delamination increases along the direction of the major stiffness of the third layer. As a consequence, the delaminated area does not have a circular shape. In Fig. 14 the plot of the damage function obtained when $\delta = 1.3$ mm is presented. In this figure the function $1-\gamma$ is plotted and a brutal damage behavior is obtained. This is essentially due to the very low value of the internal regularization parameter considered: $\varepsilon = 10^{-4}$ Nmm mm^{-2}.

The transversal displacements of the two plates constituting the laminate computed along the axes x_1 and x_2 are plotted in Figs 15 and 16, respectively. The orthotropy has a

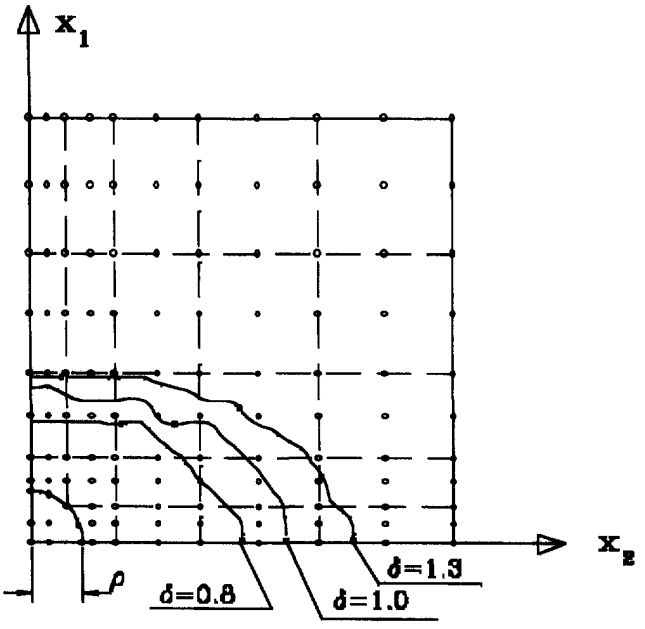

Fig. 13. Zones in adhesion and delaminated for $\delta = 0.8$, 1 and 1.3 mm.

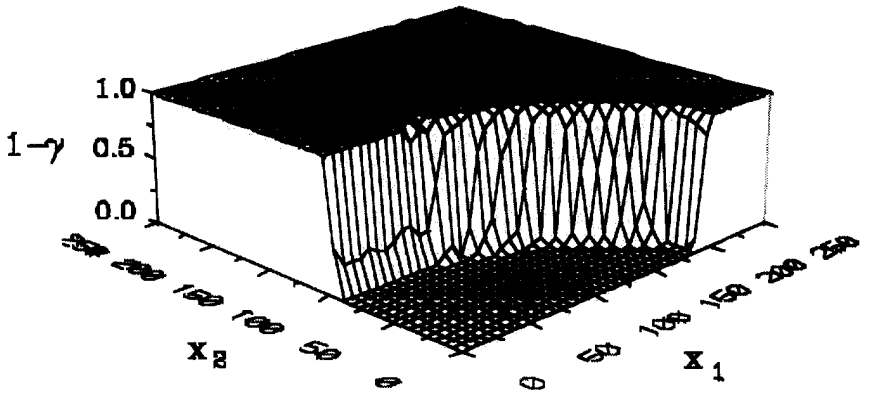

Fig. 14. Damage function at the interface for $\delta = 1.3$ mm.

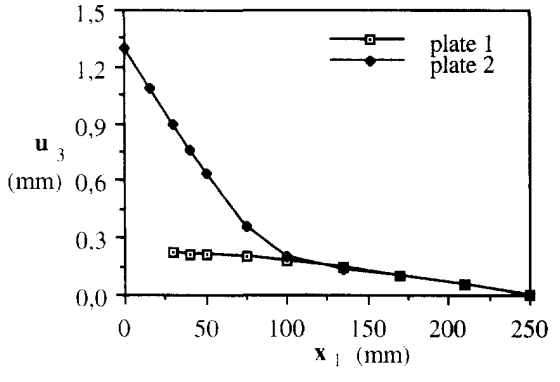

Fig. 15. Deformation of the laminate along the x_1-direction.

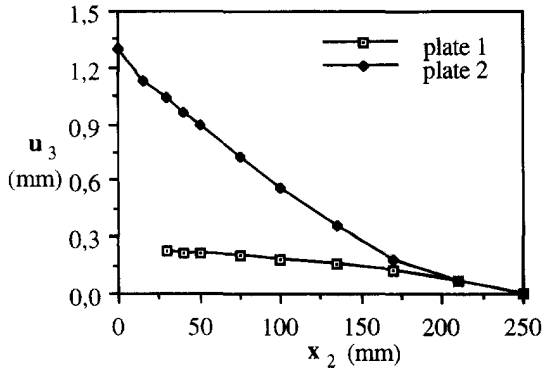

Fig. 16. Deformation of the laminate along the x_2-direction.

very strong influence in the shape of the delaminated area. Along the x_2-axis the two plates are completely independent and the delamination is large. On the contrary, along the x_1-axis the plates constituting the laminate are largely bonded.

11. CONCLUSIONS

A consistent constitutive law for the interface material, able to handle the delamination phenomenon, has been obtained starting from the Frémond's adhesion model. The proposed delamination model has been developed in the formal framework of thermodynamics. It is based on the very simple physical idea of the behavior of the interface. In fact, it has been assumed that when two points on the surfaces in contact are in adhesion, their relative displacement must be zero. In such a way, a non-smooth behavior of the interface has been obtained. Physical intuition allows one to suppose a close connection of the present model with fracture mechanics theory. The main feature of the presented delamination approach consists in the possibility of mathematically recovering fracture mechanics theory and, furthermore, to generate various regularized models with different mechanical meaning. A numerical procedure has been presented and used for beam and plate problems. Concerning the solution of the beam problem, the results obtained using the finite element method have shown a very good agreement with the analytical solution even when a fine mesh has been adopted. The problem of the numerical integration for the determination of the interface stiffness matrix has been discussed. The Simpson's integration rule with large number of integration points has been used. Numerical results related to the interesting technical problem of the delamination occurring during the drilling of a composite laminate has been presented. They have shown the delamination growth and the decrease of stiffness of the structure.

Although some comparison with experimental results appear necessary to validate the proposed approach, they are not presented herein, because the first aim of this research has been the presentation of a comprehensive approach to the modelization of the delamination phenomenon and the implementation of a robust numerical procedure. At this stage, it can be concluded that the present approach appears general enough and consistent with the thermodynamics. Furthermore, the particular regularized models and the numerical procedure proposed are simple and effective for the analysis of the delamination of composite laminates.

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